Calculus, Better Explained: Quiz

This quiz will help you check your calculus intuition. Our goal is a long-lasting perspective shift, not a collection of trivia to memorize and forget.

Read the quiz one page at a time, flipping to the next page for the answer. Enjoy!

Part 1: Intuition and Natural Description

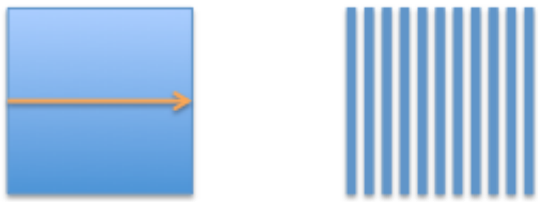
Let's start with a humble square.

This square represents a scenario: a todo list, your backyard to mow, the total distance you'll run while training for a marathon. It's a generalized notion of "stuff", like 3 is a nameless count of "something". It's up to us to say what that "something" is: 3 apples, 3 people, 3 dollars, etc.

In a similar way, the square represents the nameless pattern we have, and Calculus gives us options for splitting it apart.

Q1. How would you split the square into identical parts?

A1: To split a square into identical parts, we can do a horizontal (or vertical) split:



The yellow line indicates the starting and stopping points of our split, along with the path we take as we go. We can be more specific than "here's an equal split". Instead, "here's an equal split, and the path we used to make it".

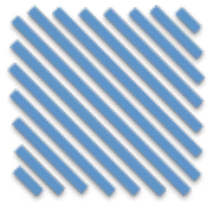
Q2. Can you split the square into parts that get increasingly larger?

A2: For parts that get increasingly bigger, we could cut "square rings" from the center:

In this scenario, the yellow line starts in the center (the first "ring") and moves to the outside. Because a ring covers all 4 sides, we only need to move from the center out.

Q3. Can you split it into parts that start small, get bigger, then get smaller again?

A3: Parts that start small, get bigger, then smaller again is tricky. But how about this:



The pieces start small in the bottom corner, gradually get larger (in a direct, linear fashion), then get smaller again.

We don't always have to slice along a purely horizontal or vertical axis. In most classes, we do, because motion directly along the x- or y-axis is easier to describe. But don't let difficulty of writing an equation limit your thinking.

Q4: Imagine the specific scenario your square represents (e.g., your todo list). Which splitting approach would you use?

A4: Math understanding should be a tool for *thinking*, not just a tool for calculating.

If the square represents the work I had to do, I'd probably take the diagonal approach. Start with an easy task to warm up, build the momentum to handle more difficult tasks, then wind down with some easy ones. Our memory is strongly influenced by the end of an event, so finishing on a good note means the memory of working on the project is fairly pleasant.

Of course, we can still *try* to work through the calculation aspect of the scenario :).

Part 2: Official Language and Calculation

Q5: What is the "math english" version of the strategies? It should be something like `derive the area of a square with respect to [X]`

A5: In each case, we are splitting the area of the square into smaller parts as we move along a path. The names are up to us, but I'd use something like this:

- `derive the area of a square with respect to the x-axis`
- `derive the area of a square with respect to the perimeter`
- `derive the area of a square with respect to the diagonal`

Q6: Find the formula for the area of a square in terms of the parts you've identified in Q5 (the x-axis, the perimeter, the diagonal).

A6: The area of a square is:

$$\text{Area} = \text{side}^2$$

Our goal is to express this in terms of the path we're taking in each of the scenarios.

Area in terms of x-axis

Our x-coordinate goes from 0 to s. The area of a square is when have gone "x" units along is:

$$\text{Area} = x \cdot s$$

Note, we could allow x to grow larger than s, which would build a rectangle. (The math is just telling us what would happen.)

Area in terms of perimeter

Since perimeter = 4 * side, we can write

$$\text{perimeter} = 4 \ \cdot \ \text{side}$$
$$\text{side} = \frac{\text{perimeter}}{4}$$

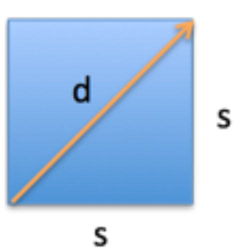
which means

$$\text{Area} = \text{side}^2 = (\frac{\text{perimeter}}{4})^2 = \frac{1}{16}\text{perimeter}^2$$

Area in terms of the diagonal

As noted earlier, this scenario is trickier. Let's work through it step by step.

First, by the pythagorean theorem, the full length of the diagonal is

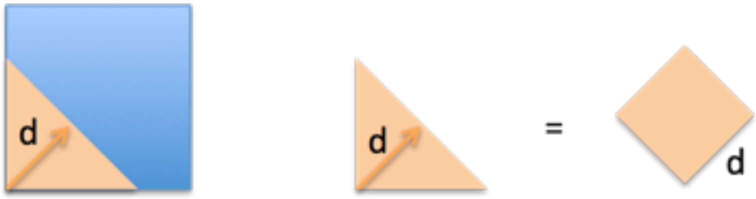


$$d^2 = s^2 + s^2$$
$$d^2 = \sqrt{2s^2}$$
$$d^2 = \sqrt{2} \cdot s = 1.414s$$

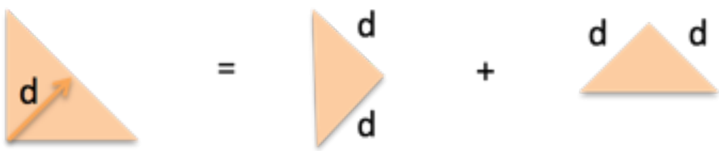
Our arrow starts at d=0 and increases until d = 1.414 times the length of one side.

Ok. How much area do we have as the diagonal grows? It's easier to split the problem into two parts.

If we're less than halfway (d < .707 * s), we have a single triangle to worry about. Geometrically, we can see the size of this triangle is d^2:



We can find this algebraically too. The diagonal splits the orange triangle into two equal parts:



$$\text{Area} = \text{Triangle1} + \text{Triangle2} = \frac{1}{2}\text{base} \cdot \text{height} + \frac{1}{2}\text{base} \cdot \text{height}$$
$$= \frac{1}{2}d \cdot d + \frac{1}{2}d \cdot d = \frac{1}{2}d^2 + \frac{1}{2}d^2 = d^2$$

If we're more than halfway (d >= .707 * s), take the area of the full square and subtract the portion we haven't yet covered.

The full square is s^2. The entire length is 1.414*s, the current diagonal is d, so the amount left to go is (1.414s - d). By similar reasoning, that triangle has area (1.414s - d)^2.

Phew! Our rules for the area in terms of the diagonal:

$$\text{Area} = d^2 \quad (\text{if d} < .707\text{s})$$
$$\text{Area} = s^2 - (1.414s - d)^2 \quad (\text{if d} >= .707\text{s})$$

Q7: Enter each formula into Wolfram Alpha. What does the result mean?

A7: Let's let the computer do the work for us.

x-axis

In the first case, we have

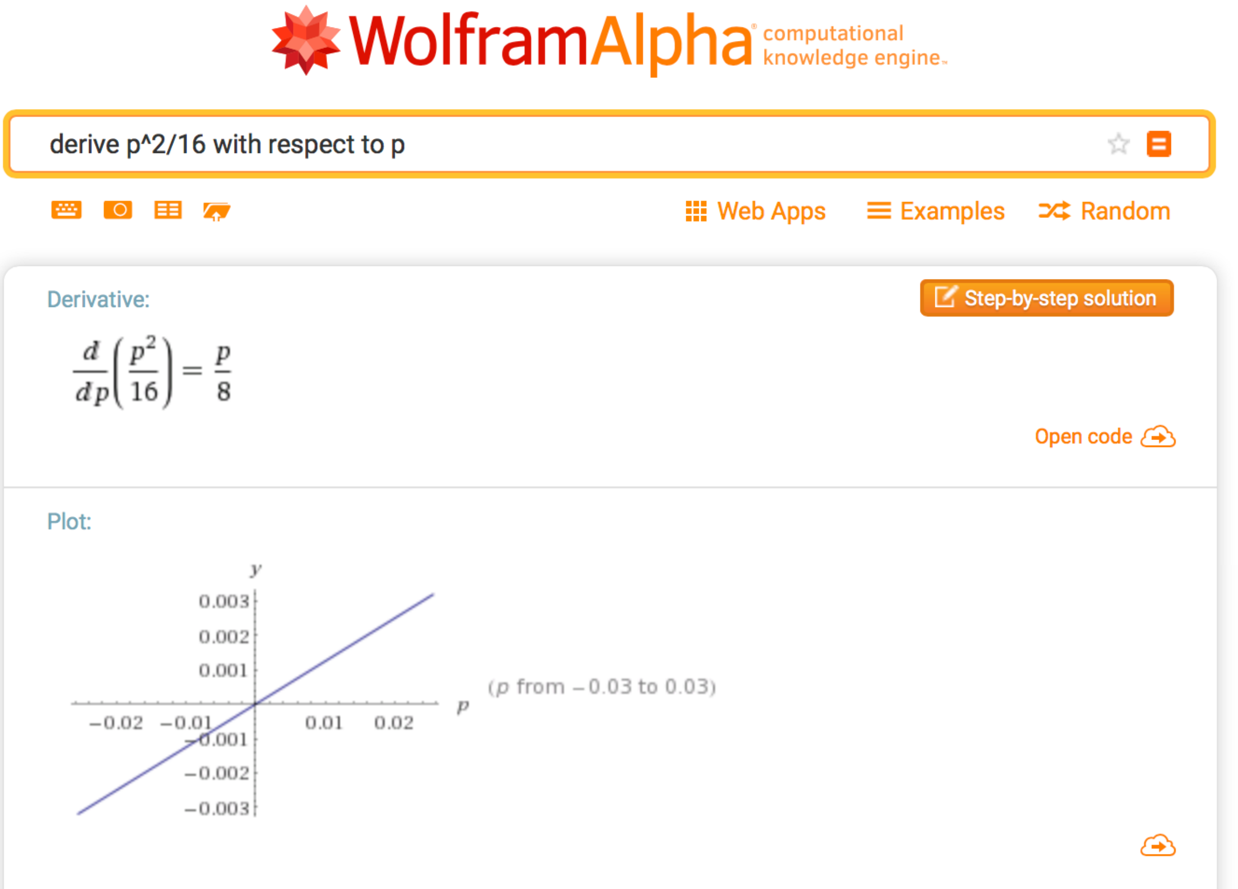
`derive x*s with respect to x`

The result is `s`. What does it mean? Every slice is equally large. `s` is a value we provide up front, the size of our square. If s = 3, every slice will have a height of 3.

Perimeter

Here our formula is:

`derive p^2/16 with respect to p`



The derivative is `p/8`. What does this mean?

For every full unit we increase the perimeter (from p=8 to p=9, for example), the area will only nudge up by p/8, where p is the current perimeter. Let's take a look.

When s=2, the area is 4 (s^2), and the perimeter is 8 (s * 4). If we increase the perimeter by 1 unit (from p=8 to p=9), it means we had to increase each side from 2 to 2.25 (2.25 * 4 = 9). Put another way, the 1 unit increase in perimeter was split among 4 sides, each gaining .25 units.

Now, what happened to the area?

```
old area = 2^2      = 4
new area = 2.25^2   = 5.0625
change in area      = 1.0625
```

Now, could we have predicted this change in area? Calculus helps. We can predict a change of

$$\frac{p}{8} = \frac{8}{8} = 1$$

However, the true increase in area is a bit more than 1 (1.0625) because our change happene din a giant leap from p=8 to p=9. If we'd moved from p=8 to p=8.1, then p=8.1 to p=8.2, the measurement would have been more accurate. Calculus gave us a rate that held for an instant, good for an estimate, but for true accuracy we have to recompate as we go.

For the exact calculation, we would integrate ("add up") our changes over the period we needed:

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integrate p/8 from p=8 to p=9
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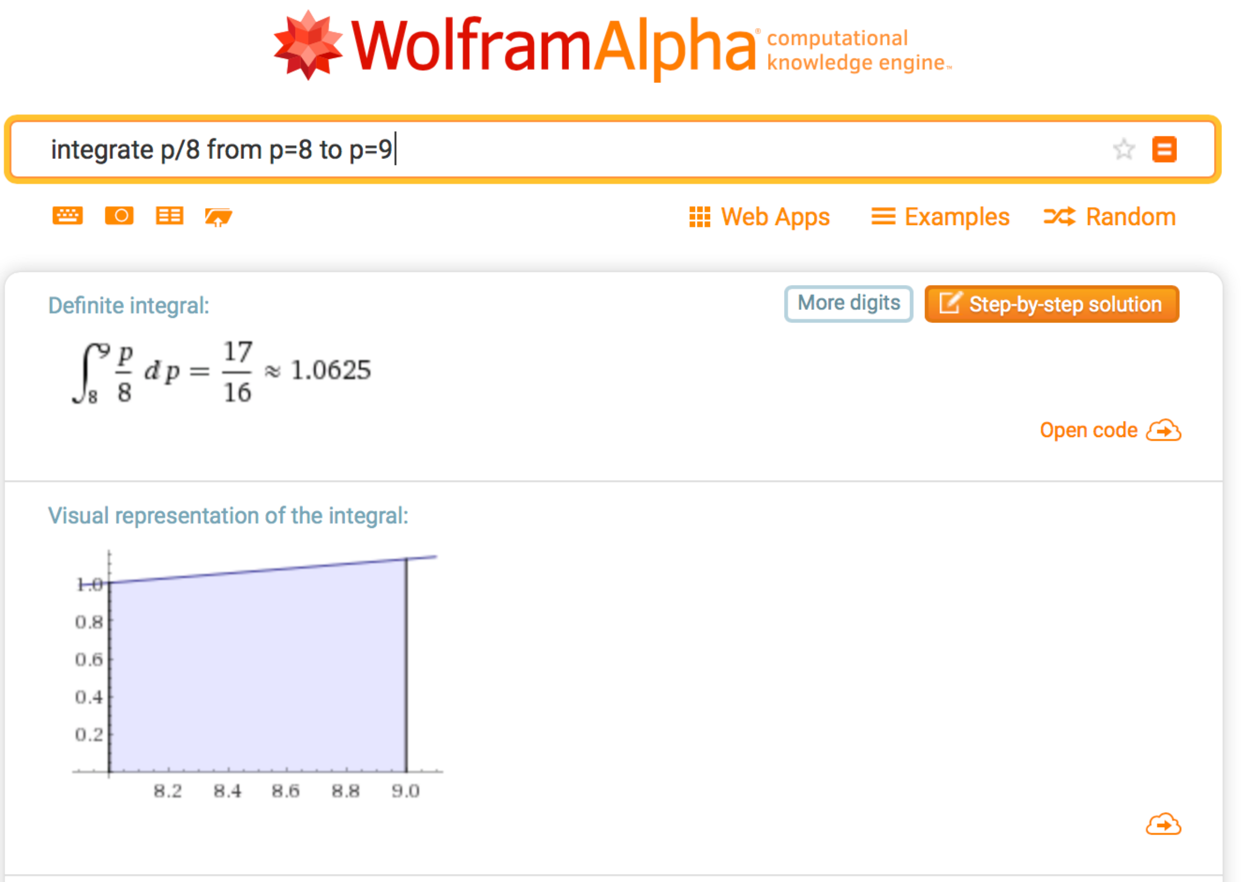

Diagonal

Here, we have two patterns to describe our derivative:

Up to the halfway point

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derive d^2 with respect to d
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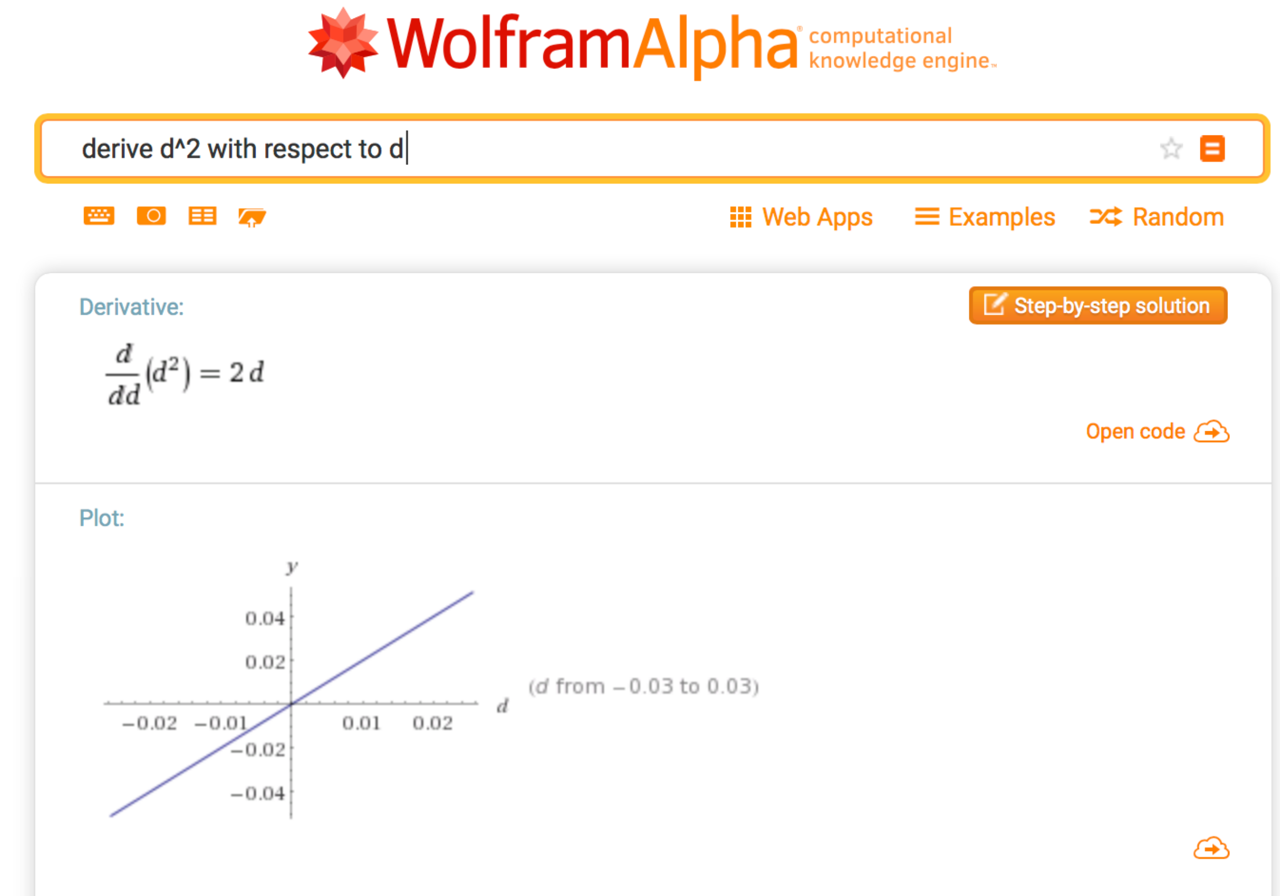
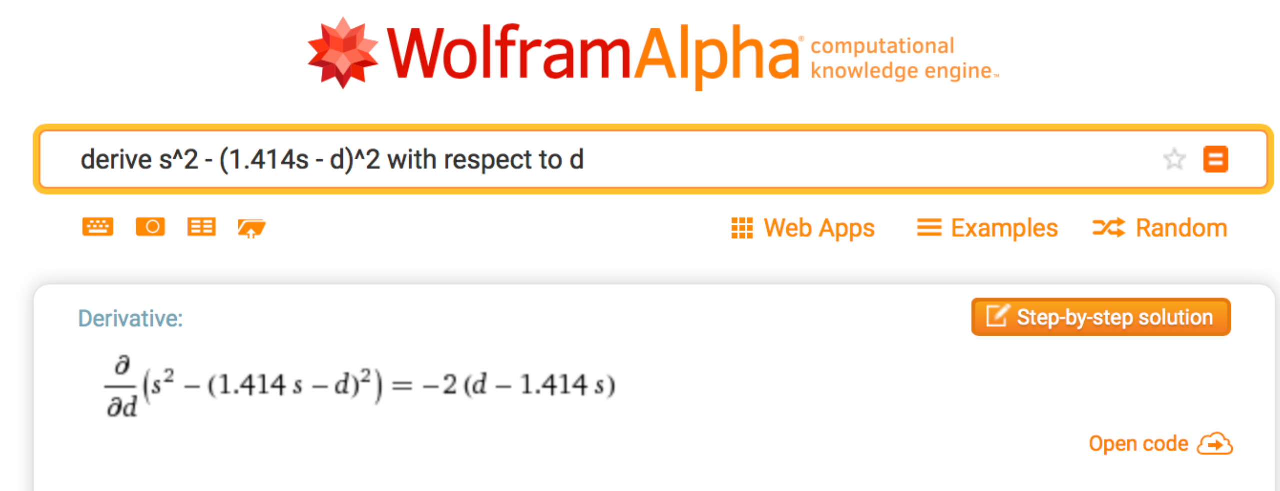


This derivative (2d) is positive and increasing. As our diagonal grows to the halfway point, we get a larger and larger change. When d=0.3, our area changes by 2 * d or 0.6. When d=.5, our area changes by 2*d=1.0.

Past the halfway point

`derive s^2 - (1.414s - d)^2 with respect to d`



We can rearrange this to:

$$2(1.414s - d)$$

This starts large and decreases as d grows. As we get further along in the diagonal, our changes slow down until they cease (when d= 1.414 * s, the full amount).

Q8: When does each pattern get to 20% of the total area? 50%? 80%?

A8: Because each pattern takes a different approach, we should expect different times to reach a given area. (Of course, each path starts at 0% and finishes at 100%.)

An easy way to solve each formula is to take a side of 1 (s=1), which means area = 1^2 = 1. Then we can solve for when we reach a = 0.2, a = 0.5, and a = 0.8.

x-axis:

Area = x * s = x [assuming s = 1]

- Area = 0.2 when x = 0.2
- Area = 0.5 when x = 0.5
- Area = 0.8 when x = 0.8

This shouldn't be too surprising -- we're taking a linear walk through the square, and should expect linear results.

Perimeter:

$$\text{area} = \frac{1}{16}p^2$$
$$p^2 = 16 \cdot \text{area}$$
$$p = \sqrt{16 \cdot \text{area}} = 4\sqrt{\text{area}}$$

- To reach area = 0.2, p = 4 * sqrt(0.2) = 1.79
- To reach area = 0.5, we need p = 4 * sqrt(0.5) = 2.83
- To reach area = 0.8, we need p = 4 * sqrt(0.8) = 3.58

The pattern changes differently. Going from 20% to 50% area required a change of 2.83 - 1.79 = 1.04 in perimeter. But going from 50% to 80% only required a perimeter change of 3.58 - 2.83 = 0.75. As we go along, we get more and more "oomph" for a change in p.

Diagonal

Let's analyze our fancy-pants scenario.

To get Area = 20%, we're less than halfway, so can solve:

$$\text{area} = d^2$$
$$0.20 = d^2$$
$$d = \sqrt{0.20} = 0.447$$

We get 20% of our area when d = .447, which is .447 / 1.414 = 31.6% of the diagonal.

To get Area = 50%, we are at the halfway point: 1.414 / 2, or .707. Moving to 50% of the diagonal gets us 50% of the area.

To get Area = 80%, we solve the trickier equation:

$$\text{area} = s^2 - (1.414s - d)^2$$
$$.80 = 1^2 - (1.414 \cdot 1 - d)^2 \quad [\text{let s} = 1\text{, solve for area} = 80\%]$$
$$(1.414 - d)^2 = 1 - .8$$
$$(1.414 - d)^2 = 0.2$$
$$1.414 - d = \sqrt{0.2}$$
$$-d = \sqrt{0.2} - 1.414$$
$$d = 1.414 - \sqrt{0.2} = .967$$

Here, we get to 80% area when d = .967. Because our gains are so large in the middle, we only need to nudge a little higher past the halfway point (.707) to reach 80%. Indeed, this is only .967 / 1.414 = 68.3% of the entire diagonal.

Conclusion

I hope this helps build a feel for calculus:

- Start with an idea of what to do
- Describe it in everyday language
- Move to technical language for a computer
- Interpet the results
- Experiment with various calculations

Over time, you'll be able to write "sheet music" directly. But never feel bad about humming to start.

Happy math.